

# What Are Virtual Manipulatives?

**A**s a result of innovations in technology, the prevalence of the Internet, and the increasing availability of computers in classrooms and homes, an enhanced approach for teaching and learning mathematics using manipulatives and computers is emerging. This new approach essentially creates a new class of manipulatives, called *virtual manipulatives*, as well as new capabilities, or toolkits, for computer programs that use visual representations. These new virtual manipulatives have all the useful properties of existing computer manipulatives while overcoming many of their disadvantages, yet very little is known or written about them. The purpose of this article is to establish a working definition of virtual manipulatives, highlight examples of virtual manipulatives on the Internet, and discuss their current and potential classroom use.

## Defining Virtual Manipulatives

Two types of representations on the World Wide Web are being called virtual manipulatives—these are *static* and *dynamic* visual representations of concrete manipulatives (Spicer 2000). Consequently, two types of virtual manipulatives are available as teaching and learning tools. Because one of these types is far more powerful and has much greater utility and potential for teaching, drawing the distinction between the two is important. Establish-

ing a name and definition that uniquely describe these dynamic images is equally important to avoid confusion.

Static visual representations are essentially pictures. They are the sorts of visual images ordinarily associated with pictures in books, drawings on an overhead projector, sketches on a chalkboard, and so on. Although such representations resemble concrete manipulatives, they cannot be used in the same ways that concrete manipulatives can. That is, a student can actually slide, flip, and turn concrete manipulatives but cannot perform the same actions with a static picture of the concrete manipulative. These static visual representations are not true virtual manipulatives.

In contrast, dynamic visual representations of concrete manipulatives are essentially “objects.” They are visual images on the computer that are just like pictures in books, drawings on an overhead projector, sketches on a chalkboard, and so on. In addition, these dynamic visual representations can be manipulated in the same ways that a concrete manipulative can. Just as a student can

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slide, flip, and turn a concrete manipulative by hand, he or she can use a computer mouse to actually slide, flip, and turn the dynamic visual representation as if it were a three-dimensional object. This kind of visual representation is truly a virtual manipulative.

Although many people currently use the term *virtual manipulative* to refer to any computer-generated image that appears on a monitor and is intended to represent concrete manipulatives, this meaning is much too simplistic to be descriptive or useful. The ability to manipulate the visual representation, or object, on the computer connects the user with the real teaching and learning power of virtual manipulatives, that is, the opportunity to make meaning and see relationships as a result of one's own actions. This kind of user engagement distinguishes virtual manipulative sites from those sites where the act of pointing and clicking results in the computer's providing an answer in visual or symbolic form. We think that the term *virtual manipulative* should be restricted to describing this interactive capability.

In our view, a virtual manipulative is best defined as an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge. Currently, virtual manipulatives are modeled on the concrete manipulatives commonly used in schools, such as pattern blocks, tangrams, fraction bars, geoboards, and geometric solids. When such objects are available through the Web, they may be considered virtual. However, their ability to be used interactively—that is, to allow the user to engage and control the physical actions of these objects—combined with the opportunities that they offer to discover and construct mathematical principles and relationships, distinguishes them as virtual manipulatives. With current technology, the most common way of effecting these movements on a monitor is by using a computer mouse to point, click, and drag. The future will undoubtedly bring many other ways of moving virtual manipulatives, perhaps through such features as voice commands and infrared signals.

## Virtual Manipulatives and the Merely “Virtual”

As previously mentioned, distinct differences in interactivity exist among Web sites that use virtual manipulatives. On the one hand are sites that limit user involvement by displaying mathematical concepts as static images on the monitor. On the other hand are dynamic sites that let users extensively manipulate and even alter the representations to explore concepts and construct meaning. Teachers

who would provide interactive opportunities for their students to construct mathematical knowledge must learn to distinguish between what is truly a virtual manipulative and what is merely “virtual.”

### Static images or representations

Virtual sites consist of static images or representations. An example of such a site is *Visualize Fractions* at [www.visualfractions.com](http://www.visualfractions.com). This site presents a number line and circular representations of fractions; users answer questions using the proper symbolic notation. The images change on the screen to prompt responses, but the user cannot move or manipulate the image. The site does not offer interactive opportunities for users to construct meaning on their own.

### Computer-manipulated images or representations

Some sites present a more representational view but still do not allow users to manipulate images. For example, the computer might move the images in response to a command from the user, but the user does not directly manipulate the object. Again, because these sites preclude user involvement, they cannot be considered to incorporate true virtual manipulatives. One of these sites is *Room 108* at [www.netrover.com/~kingskid/MulTab/Applet.html](http://www.netrover.com/~kingskid/MulTab/Applet.html), which provides an applet for multiplication in its “Math” section. Numbers are arranged in a multiplication table, and users click on two factors and the equals sign (e.g.,  $2 \times 3 =$ ); the computer then displays the answer as both a  $2 \times 3$  rectangular array of squares and as the numeral 6. The program does not allow the user to take an active role in creating either representation. The user cannot actually move the object and, through that movement, create mathematical meaning.

### Virtual manipulative Web sites

Web sites that incorporate virtual manipulatives offer much more interactive experiences for users. *Manipula Math with JAVA* ([www.ies.co.jp/math/java/index.html](http://www.ies.co.jp/math/java/index.html)), for example, allows users in the middle school section to choose from several applets that demonstrate and verify the Pythagorean theorem. In these demonstrations, the user is presented with a right triangle to adjust. The theorem is investigated by moving pieces of the squares that represent  $a^2$  and  $b^2$  and placing them into the area that represents  $c^2$ . The various applets

**Students can slide, flip, and turn a visual representation as if it were a three-dimensional object**

use different methods for fitting the pieces into  $c^2$ . Another site that offers a similar experience is the “Pythagoras’ Theorem” page at [www.keypress.com/sketchpad/java\\_gsp/PIT.HTM](http://www.keypress.com/sketchpad/java_gsp/PIT.HTM).

David E. Joyce’s “The Geometry Applet,” version 2.2 ([aleph0.clarku.edu/~djoyce/java/Geometry/Geometry.html](http://aleph0.clarku.edu/~djoyce/java/Geometry/Geometry.html)) and Paul Flavin’s “Java 3d Viewer” ([www.frontiernet.net/~imaging/java-3d-engine.html](http://www.frontiernet.net/~imaging/java-3d-engine.html)) offer dynamic experiences in three-dimensional geometry. Texas A & M University’s two “Algebra Tiles” pages ([www.coe.tamu.edu/~strader/Mathematics/Algebra/AlgebraTiles/AlgebraTiles1.html](http://www.coe.tamu.edu/~strader/Mathematics/Algebra/AlgebraTiles/AlgebraTiles1.html) and [/AlgebraTiles2.html](http://www.coe.tamu.edu/~strader/Mathematics/Algebra/AlgebraTiles/AlgebraTiles2.html)) also offer users more dynamic experiences for exploring concepts in algebra. Here, users arrange the tiles in a rectangle with dimensions that represent the factors of a given quadratic expression.

Dynamic sites that provide physical manipulatives in virtual form for teachers and students to control are true virtual manipulative sites. Along with the virtual toolkits, these sites offer descriptions of their potential uses for mathematics instruction and practical instructions on how to

manipulate the virtual forms. Users have the flexibility to use the virtual manipulatives as they choose. Some of these sites include links to suggested lessons or activities, but their use is not limited to these suggestions. One example of this type of site is Jacobo Bulaevsky’s *Arcytech* at [www.arcytech.org/java/](http://www.arcytech.org/java/).

By clicking on one of the four available tools, the user links to a description of the tool, instructions on how to use it, and a list of prepared lessons. For example, the pattern-block program includes links to Cynthia Lanius’s site, called “No Matter What Shape Your Fractions Are In” ([math.rice.edu/~lanius/Patterns/index.html](http://math.rice.edu/~lanius/Patterns/index.html)), which provides lessons and activities for teaching and learning fractions using pattern blocks. In addition, Bulaevsky’s base-ten program links to Margo Mankus’s Web page, “Base Ten Block Activities” ([mason.gmu.edu/%7Emmankus/whole/base10/baseten.htm](http://mason.gmu.edu/%7Emmankus/whole/base10/baseten.htm)), which provides a lesson plan for using base-ten blocks in teaching place value and arithmetic operations.

The best virtual manipulative sites are those with a variety of dynamic features that allow users to perform various mathematical investigations. One of the most extensive collections of virtual manipulatives currently on the Web is Utah State University’s *National Library of Virtual Manipula-*

*tives* ([matti.usu.edu/nlvm/index.html](http://matti.usu.edu/nlvm/index.html)). This site, funded by the National Science Foundation, features Web-based virtual manipulatives or concept tutorials, primarily in the form of Java applets, for mathematics instruction, mainly at the elementary and middle school levels ([matti.usu.edu/nlvm/nav-asp/projectinfo.html](http://matti.usu.edu/nlvm/nav-asp/projectinfo.html)). Users can choose from the following strands: numbers and operations, algebra, geometry, measurement, and data analysis and probability. The site includes instructions on using the manipulatives, activities to accompany the site, and links to the NCTM’s Standards (NCTM 2000). Researchers developing this site have several important goals, including interactivity for the user, the user’s control of variable aspects, and the opportunity to explore and discover mathematical principles and relationships (Cannon, Heal, and Wellman 2000).

Another site for which various types of manipulative tools are being developed is George Mason University’s *Dr. Super’s Virtual Math Manipulatives Project* ([www.galaxy.gmu.edu/~drsUPER/](http://www.galaxy.gmu.edu/~drsUPER/)). The goal of this project is to create and develop curriculum materials and technology-based products for various new physical manipulatives, including Triange, Dr. Super’s Triangles, Super Math, Trigram, Factor Blocks, Geoshapes, and GeoJunior. The project has two components. One will give educators virtual toolkits with which they can develop activities and lessons as needed; the other will offer ready-made lessons and activities using the toolkits. Some sample activities for these developing experiences can be found on the Web site.

The NCTM recently launched an electronic version of its *Principles and Standards for School Mathematics* (NCTM 2000). This site includes electronic examples under the “Standards” page and “I-Math Investigations” (in the *Illuminations* section of “Teacher Resources”). The interactive tasks are organized by the grade bands K–2, 3–5, 6–8, and 9–12 and are keyed to specific Standards ([standards.nctm.org](http://standards.nctm.org)). Several of these examples include virtual manipulative experiences for users.

## Unique Properties of Virtual Manipulatives for Classroom Use

Beyond the variances in dynamic capabilities, the most important issue to consider in discussing virtual manipulatives is how they are used in classrooms to support students’ understanding of mathematics concepts. Extensive use and research will eventually help us discern the most effective uses of these tools. Given what is already known about virtual manipulatives and virtual manipulative Web

**The problem of not having enough blocks is controlled by the click of a mouse**

sites, we can propose some properties that make them particularly advantageous for classroom use.

### Advantageous properties

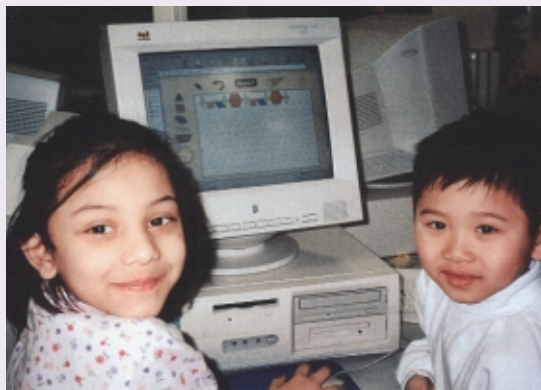
Virtual manipulatives promise to be great assets for classroom use. Already they can link iconic and symbolic notations, highlight important aspects or features of individual manipulatives, offer links to other resources on the World Wide Web, and record and store users' movements and other work tasks. They also foster as much engagement as physical manipulatives do (Dorward and Heal 1999).

Although computer programs already provide interactive manipulative experiences for students, the advantage of many emergent virtual manipulatives is that they are on the Web, thereby allowing free access for schools that are online and constant availability for busy teachers and students who have limited time to get to these sites during class. With a Web connection, these interactive sites are available anywhere, anytime, to anyone. Parents and students can use these virtual manipulatives from their home computers. Teachers who may be reluctant to send home concrete manipulatives for students' use may be more likely to give assignments to students who have access to virtual manipulatives through their home computers. Availability is just one of several factors that teachers might consider when they determine the effectiveness of using virtual manipulatives in their classrooms. (See **fig. 1** for a list of possible evaluation questions.)

Another unique aspect of virtual manipulatives is their potential for alteration. For example, because some virtual manipulative sites allow users to color parts of the visual object, students can add color for one-to-one correspondence when counting objects. Other sites allow students to add lines or points to figures, a feature that would be

**FIGURE 2**

**Kindergartners display their representations of repeating patterns using virtual pattern blocks**



Photograph by Patricia S. Moyer and Johnna J. Boyard; all rights reserved

useful for counting or marking the sides of a polygon to determine its shape or perimeter. Students can also mark edges, vertices, and faces to highlight these aspects on Platonic solids. (See, for example, [www.illuminations.nctm.org/imath/3-5/GeometricSolids/GeoSolids1.html](http://www.illuminations.nctm.org/imath/3-5/GeometricSolids/GeoSolids1.html).)

Teachers sometimes express concern that they do not have enough concrete manipulatives for all students in their classes and that distribution and cleanup of materials is time-consuming. With virtual manipulatives, students control the number of blocks available with the click of a computer mouse. Cleanup is as fast as clicking the mouse on a recycling bin or a broom icon. (See, for example, the base-ten blocks at [www.arcytech.org/java/b10blocks/b10blocks.html](http://www.arcytech.org/java/b10blocks/b10blocks.html).)

Another consideration in using any type of manipulative, particularly with students in the upper elementary and middle school grades, is the notion that manipulatives, such as patterns blocks and tangrams, are "just for little kids." For students in grades 4 through 8, the use of virtual manipula-

**FIGURE 1**

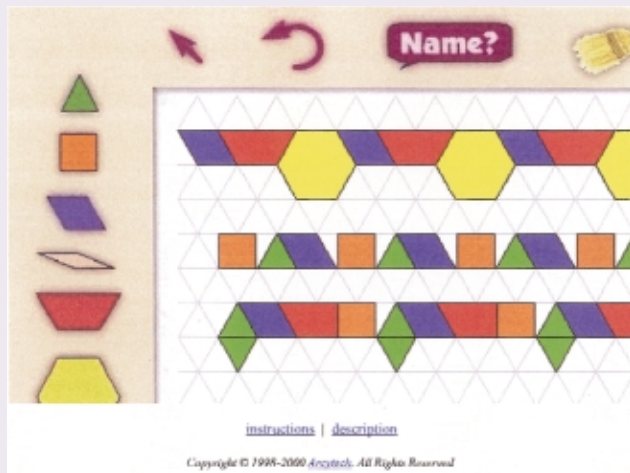
#### Important questions for evaluating a virtual manipulative site

- Is the site easy to find?
- Are parts of the site easy to use? Are the procedures clear for rotating, flipping, and otherwise manipulating the objects used in the site?
- Are the instructions clear on every page? Can the user quickly figure out how to manipulate the objects on the site?
- Are the images dynamic and interesting? Do they represent the targeted mathematics and allow the user to make a connection between the concrete or virtual pictorial and the symbolic?
- What type of learning can be achieved using the site? Does this site offer suggestions for activities, or is the educational experience self-evident? Remember that drill and practice are important but are not appropriate when students are trying to construct knowledge of a concept.
- Does the site offer flexibility? Can the tool be used to teach several different concepts? Does it allow users or teachers to design their own problems?
- If questions are asked, is some type of constructive feedback presented? For example, rather than allow the user to keep guessing indefinitely, the site should offer periodic helpful prompts.



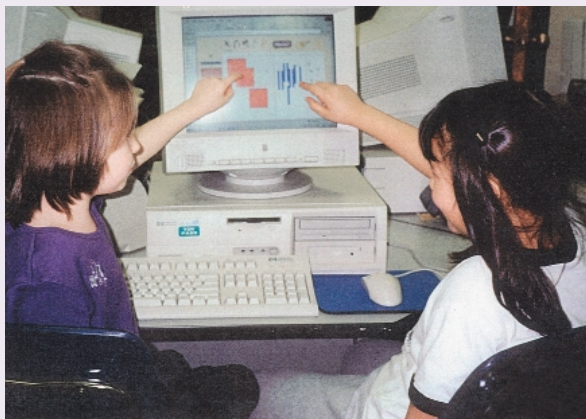
**FIGURE 3**

**Printed copies of kindergarten's repeating patterns**



**FIGURE 4**

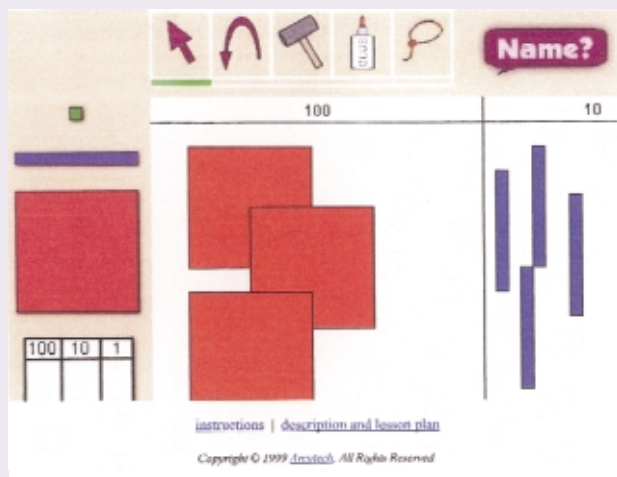
**Second graders' representation of 394 using virtual base-ten blocks**



Photograph by Patricia S. Moyer and Johnna J. Bolyard; all rights reserved

**FIGURE 5**

**Virtual base-ten block applet showing the mallet icon**



tives may remove the connotation of “playing with blocks.” Older students may view the use of virtual manipulatives on the computer as more sophisticated than using manipulatives in their concrete form.

### Current use

Classroom teachers are just beginning to experiment with the capabilities of using virtual manipulative sites for instruction. Debbie Niezgoda and John Stanley used a virtual manipulative site with their kindergarten classes at Westlawn Elementary School in Falls Church, Virginia, to reinforce concepts of repeating patterns. During one lesson, the kindergartners used virtual pattern blocks (found at [www.arcytech.org/java/patterns](http://www.arcytech.org/java/patterns)) to create patterns on the computer screen (see **fig. 2**). The children developed complex repeating patterns and were not limited by the number of blocks available, because the pattern-block applet allowed the children to generate as many copies of the blocks as they wanted. The children then printed color copies of their repeating patterns (see **fig. 3**). Using the virtual manipulatives enabled the children to generate as many pattern blocks as they needed to create their designs in both horizontal and vertical formats. The abilities to generate as many blocks as needed and to print out hard copies of the completed designs are both advantages of using virtual blocks over concrete pattern blocks.

The children in Charlotte Davnie’s second-grade classroom at the same school used virtual manipulative base-ten blocks (found at [www.arcytech.org/java/b10blocks](http://www.arcytech.org/java/b10blocks)) to demonstrate their understanding of place value in representational form. Students showed representations of a three-digit number, such as 394, using the virtual base-ten blocks (see **fig. 4**). The mallet icon on this applet allows children to break the hundreds and tens representations into tens and ones, respectively (see **fig. 5**). This feature allows children to simulate the “take-away” model for subtraction, whereas the concrete base-ten blocks do not.

In Johnna Bolyard’s middle school classroom at Farmwell Station Middle School in Ashburn, Virginia, eighth-grade students used virtual Platonic solids (found at [illuminations.nctm.org/imath/3-5/GeometricSolids/index.html](http://illuminations.nctm.org/imath/3-5/GeometricSolids/index.html)) to assist them in designing versions of a dodecahedron, tetrahedron, cube, octahedron, and icosahedron. The virtual versions of the solids could be rotated and viewed, allowing the students to compare different solids, count their faces, and identify their shapes as they constructed straw models (see **fig. 6**). In past lessons, when Bolyard used static pictures of the solids, the students had difficulty translating the

two-dimensional view into the three-dimensional solids. Using the virtual manipulative versions of the solids enabled the students to construct their own understandings about the number and shapes of the faces on each solid and the number of faces that met at a vertex. The ability to see all the faces gave the students a better understanding of the construction of the solid.

As virtual manipulatives are enhanced and become more available, the possibilities for use are limited only by the creativity of the teachers and students who work with them. One important contribution that teachers can make to the knowledge base about virtual manipulatives is to use these sites during instruction in their classrooms and share their insights and experiences with others through how-to articles and research reports.

## Closing

In this article, we have presented a basic introduction to the world of virtual manipulatives, but much of this area still remains to be seen and explored. Work on virtual manipulative sites is still under development at various universities, and by the time this article appears in print, numerous new virtual manipulative sites may be developed and launched. As technology continues to evolve in this area, the possibilities for what might be created in the next few years are exponential. Along with new ways of manipulating visual representations, entirely new forms of representation may emerge, including manipulatives that have no counterparts in the physical realm. Perhaps the wireless classroom, in which all students have laptops and access to their own collections of virtual manipulatives, is just around the corner. Virtual manipulatives may very well be the most appropriate mathematics tool for the next generation.

Authors' note: The authors invite readers to submit examples of virtual manipulative Web sites. We are particularly interested in hearing how you and your students use these sites and about the impact that they have on students' learning of mathematics. The authors extend special thanks to Elizabeth Milewicz for her feedback on the manuscript.

## References

- Cannon, L. O., E. R. Heal, and R. Wellman. "Serendipity in Interactive Mathematics: Virtual (Electronic) Manipulatives for Learning Elementary Mathematics." *Journal of Technology and Teacher Education, Proceedings of the Society for Information Technology and Teacher Education*. San Diego, Calif.: February 2000.
- Dorward, J., and R. Heal. "National Library of Virtual Manipulatives for Elementary and Middle Level Mathematics." *Proceedings of WebNet 99 World Conference on the WWW*

## FIGURE 6

**Eighth graders constructing straw models of the Platonic solids using virtual manipulatives**



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and Internet, pp. 1510–12. Honolulu, Hawaii: Association for the Advancement of Computing in Education, October 1999.

National Council of Teachers of Mathematics. (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: NCTM, 2000.

Spicer, J. "Virtual Manipulatives: A New Tool for Hands-On Math." *ENC Focus* 7 (4) (2000): 14–15. ▲