Three Components of Algebraic Thinking: Generalization, Equality, Unknown Quantities

For many people, the thought of studying algebra conjures up memories of "an intensive study of the last three letters of the alphabet" (Blair, 2003). While this description of algebra may be representative of most people's high school experiences with the subject, much of algebra does not involve problems with *x, y,* or *z.* In fact, the potential for students to think algebraically resides in many of the arithmetic problems they regularly do in upper elementary school; it requires only a shift in language or a slight extension of a basic arithmetic problem to open up the space of algebraic thinking for students (see, for example, Usiskin, 1997). Algebraic problems in elementary school do not have to include the dreaded phrase, "Solve for *x*."

Considering the role of algebra in grades 3 – 5 requires us to go beyond the limited definition of "problems with letters" to a more generative view of algebraic thinking. A useful definition of algebraic reasoning is given by John Van de Walle (2004), who writes: "*Algebraic reasoning* involves representing, generalizing, and formalizing patterns and regularity in all aspects of mathematics." (p. 417). Algebra is, in essence, the study of patterns and relationships; finding the value of *x* or *y* in an equation is only one way to apply algebraic thinking to a specific mathematical problem.

As we think about algebraic reasoning, it may also help to define the term *algebra*. The NCTM *Principles and Standards for School Mathematics* (2000) includes a description of algebra that goes beyond manipulating symbols. In the *Standards*, algebra is defined as:

- Understanding patterns, relations, and functions;
- Representing and analyzing mathematical situations and structures using algebraic symbols;
- Using mathematical models to represent and understand quantitative relationships and
- Analyzing change in various contexts.

In this course, we will consider three distinct aspects of algebraic thinking that can be identified in elementary mathematics instruction: generalization, concepts of equality, and thinking with unknown quantities. These three components of algebraic reasoning provide a useful framework for recognizing whether students in grades 3 through 5 are thinking algebraically, and for determining whether a problem can be viewed algebraically.

Generalization

Prominent in most definitions of algebra is the notion of "patterns." The ability to discover and replicate mathematical patterns is important throughout mathematics. The authors of the *Principles and Standards for School Mathematics* talk extensively about the important role that understanding patterns plays in algebraic thinking:

> In grades 3–5, students should investigate numerical and geometric patterns and express them mathematically in words or symbols. They should analyze the structure of the pattern and how it grows or changes, organize this information systematically, and use their analysis to develop generalizations about the mathematical relationships in the pattern. (NCTM, 2000)

Young students can have meaningful experiences with generalizing about patterns, even though they do not usually express their mathematical ideas using variables and standard functions. For example, when exploring a pattern such as 1, 3, 5, 7, 9,…, young students may make the following observations:

1. “If you add 1 to an even number, you always get an odd number”
2. “If you add 2 to an odd number, you always get another odd number”
3. “If you start at 1 and keep adding 2, you get all the odd numbers”
4. “If you can separate a number into two equal groups, it’s an even number. If one’s left over, it’s an odd number.”

All of these observations are ways of thinking about a simple pattern—the progression of positive odd integers. However, they also provide evidence of algebraic reasoning, because each description relies on some sort of generalization that can be applied to any number. For example, notice how observation 1 contains the term “an even number.” The student here is generalizing that no matter how large or small the even number, adding 1 will create an odd number. Likewise, in observation 4, the student has identified the property that any even number can be split into even groups, but odd numbers cannot. Both of these observations are examples of generalization, since they are projecting a mathematical property onto a whole category of numbers; in this case, “the even numbers.”

It may take some time for students to develop strategies for justifying a pattern. The first steps are noticing that there is a pattern in a number sequence, and then wondering if that pattern continues as the numbers get larger. Describing the pattern is the next step, followed by extending it. Eventually, students will arrive at a generalized understanding of the pattern; they will be able to predict whether a specific number (or term) is part of a pattern without calculating each consecutive term. For example, given the pattern 1, 3, 5, 7, 9, …, above, students will be able to determine that a number such as 381 is part of the pattern because it is an odd number, and will not need to write out each odd number from 1 to 381 to be convinced of this fact.

In upper elementary school, most students will be ready to work on proving statements such as “adding 2 to an odd number produces another odd number,” but their ideas about proof will continue to evolve as they expand upon them in middle school. From a formal algebraic perspective, all four statements above follow from the fact that all odd numbers are of the form $2n+1$, but students can make and test conjectures long before they ever see such an expression.

It is important to keep in mind that as students propose generalizations such as those above, they may be basing their claims on only one or two instances of a pattern. Mathematically this is not enough evidence to determine whether a pattern exists. In observation 3 above, for example, a student may have noticed that adding 1 + 2 = 3 (adding 2 to an odd number produced another odd number), and adding 3 + 2 = 5, also an odd number. But she may not have investigated any numbers beyond those. It is important for elementary students to learn that forming generalizations from only a few instances can lead to inaccurate conclusions.

One example of this can be seen in students’ solutions to the “Frog in the Well” problem in this session’s video. The students in the video try to use generalization to solve this problem, and figure that the frog climbs 2 feet total for each 2-hour period because he climbs up three feet in the first hour and slips down one in the second hour. Using this generalization, they come to the conclusion that it will take a frog 10 hours to reach the top of a 10-foot well. However, while the relationship holds in general for each 2-hour period, the ninth hour occurs in the middle of a 2-hour period. During this hour the frog reaches the top of the well and climbs out, and consequently does not “slide down.” While students have made a generalization that is true in most cases, they have neglected to notice that their current problem is an exception to the general rule of up three, down two. In the end, their understanding of the relationship actually misleads them into solving the problem incorrectly.

Equality – the meaning of the “=” sign

Elementary texts sometimes hint at the relationships between arithmetic and algebra by noting that the problem “add 5 + 24” could just as well be stated “5 + 24 = ?” or “5 + 24 = ❑” or even “5 + 24 = *x*.” While these notations create a connection between arithmetic and the “missing value” image of algebra, students can also be misled by the implications of these expressions.

For example, consider the algebraic statement “5 + 24 = ? + 15.” On the face of it, this expression is similar to the previous ones (e.g. 5 + 24 = ?) but there is one very important difference: the number that replaces the ? is no longer 29, but a smaller number that when added to 15, produces 29. However, when faced with a problem like this, research has shown that many elementary students persist in saying the answer is either the sum of the two addends to the left of the equals sign, or the sum of all the addends in the problem, regardless of their placement relative to the equals sign (Falkner, Levi, & Carpenter, 1999). Consequently, given the problem “5 + 24 = ? + 15,” most elementary students would respond that the missing number was either 29 or 44..

What’s going on here? The issue resides in the meaning students assign to the “=” sign. In the case of the problem “5 + 24 = ?” the “=” can be thought of as “the result of the previous computation.” That is a sufficient interpretation in this problem. However, in the example “5 + 24 = ? + 15,” the equals sign must be interpreted differently. It is now a statement of equivalence between two quantities, in this case between “5 + 24” and “? + 5.” Now it is clearer that the ? must be replaced by something other than 29, since “5 + 24” and “29 + 15” are not equivalent.

Understanding that the sign “=” requires that one side of the expression be equivalent to the other is a basic tenet of algebra. Students will be stretching their algebraic reasoning skills if they see a variety of problems with unknowns in different positions, such as:

- 4 + ? = 17 + 2
- ? + 15 = 12 + 32
- 13 + 24 = 50 + ? *(Note that this problem has a negative integer as a value for ?, which may or may not be appropriate for your students.)*

Thinking about unknown quantities

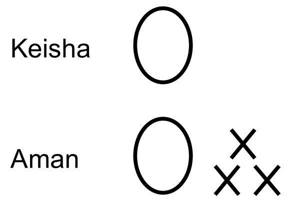
Besides the word “variable,” “unknown” is one of the words most frequently associated with algebra. Along with this concept comes the idea that the “unknown” will eventually become “known;” this is what solving equations is usually about. But it’s possible (and important) for students to work with expressions that include a variable that remains unknown. Most number tricks of the form, “choose a number, multiply it by 3, add 6, divide by 3, subtract 2 and tell me the number – and I’ll tell you your original number,” can be expressed algebraically without the need to use an specific number. The algebraic component is that the trick works for all numbers, not just a specific one for which we have to solve.

Here’s an example of a problem with an unknown quantity that remains unknown. This problem is appropriate for students in grades 3 through 5:

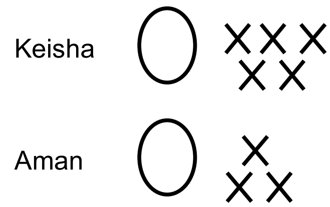
> Suppose Keisha has some number of pieces of candy in her bowl. Aman has 3 more pieces of candy than Keisha has. Keisha’s mother gives her 5 more pieces of candy. Now who has more? How many more? Then Keisha gives Aman one of her pieces of candy. Now who has more? How many more?

Students can solve this problem without creating algebraic expressions that contain variables. They may draw a picture to represent the number of candies Keisha has (e.g. a circle), and then represent Aman’s candies with a circle and three extra X’s. They could then manipulate the pictures without ever specifying what is in the circle.

In this problem, finding the exact amount of candies Keisha has is not important, since the problem asks for a comparison between two quantities.



In this diagram, the amount of candy that Keisha and Aman begin with is represented as ovals. The extra pieces the Aman has are represented as *x*'s. This diagram shows that Aman has more candy, since he has 3 *x*'s, and Keisha has none.



In this diagram, Keisha has been given 5 more pieces of candy, which are represented by *x*'s. Since she has more *x*'s (or individual pieces of candy) than Aman, she must also have more total candy, because the quantities in the ovals are the same.

However, some problems similar to the one above cannot be solved without figuring out the value of an unknown number of candies. For example:

> Suppose Keisha has some number of pieces of candy in her bowl. Aman has 3 more pieces of candy than Keisha has. If Keisha gets more candies so that she has twice as many as before, who has more candies now? How many more?

There isn't a single answer to this problem; it depends on how many candies Keisha had to begin with. So, if Keisha had 2 candies originally, Keisha will now have 4, and Aman will have 5. On the other hand, if Keisha begins with 5 candies, then she will now have 10, while Aman has 8. The difference between these two kinds of problems is subtle, but as students approach fifth grade, they should be able to start making the distinction and solving them appropriately.

These types of problems help develop algebraic reasoning skills because they require students to think flexibly about quantities, and to learn how to compare related quantities. They also promote the idea that the relationship between two quantities (here, whether Keisha or Aman has more candies) can change depending on how the original amount is acted upon.

Getting ready for formal algebra

If students have experiences with all three kinds of algebraic thinking tasks in elementary school, they will come to the "formal" study of algebra in middle and high school armed with an already-developed ability to reason algebraically. For example, as they encounter more complex linear equations in middle school, students will be able to interpret the "=" sign as an indication of equality, not as a sign requiring them to compute something. They will have already considered the kinds of patterns that they may now be asked to express in algebraic form. And they will be prepared to work flexibly with variables as unknown quantities rather than needing to figure out its value immediately. With these insights in hand, students will find that algebra is not a mystery, but a territory that already has familiar landmarks.

References

Blair, L. (2003). It's Elementary: Introducing Algebraic Thinking Before High School. *SEDL Letter, 15*(1), 16-20.

Falkner, K., Levi, L., & Carpenter, T. (1999). Children's Understanding of Equality: A Foundation for Algebra. *Teaching Children Mathematics, 6*(4), 232-236.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Usiskin, Z. (1997). Doing Algebra in Grades K-4. *Teaching Children Mathematics, 3*(6), 346-356.

Van de Walle, J. A. (2004). *Elementary and middle school mathematics: teaching developmentally (5th ed.).* New York: Longman.