1 Developing Algebraic Habits of Mind

A Framework for Classroom Questions Aimed at Understanding Student Thinking

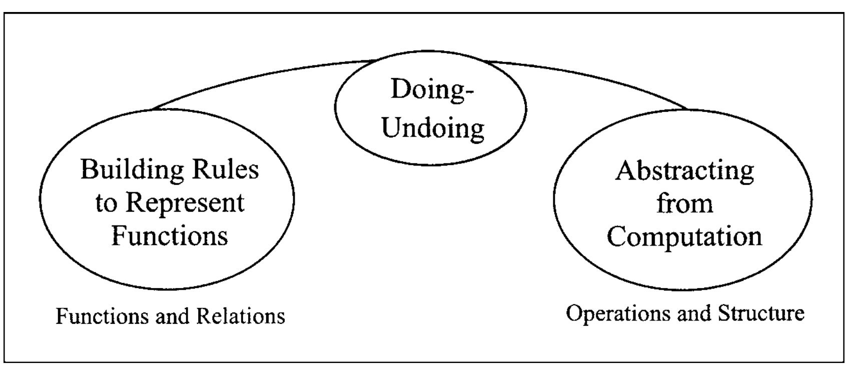
Because algebra comprises so many mathematical features, the term *algebraic thinking* defies simple definition. Generally, those who use the term do so after first choosing to concentrate on particular features, and then concern themselves with the thinking that those features demand. For example, some focus on the abstract features that distinguish algebra from arithmetic. With that perspective, they might characterize algebraic thinking as "the ability to operate on an unknown quantity as if the quantity was known, in contrast to arithmetic reasoning which involves operations on known quantities" (Langrall & Swafford 1997, 2). Others focus on the important role that functions play in algebra, and may characterize algebraic thinking as the capacity to represent quantitative situations so that relations among variables become apparent. Yet others may have problem solving as their point of reference for thinking about algebra and for thinking algebraically, and might concern themselves with how problem solvers model problems.

Our perspective has been influenced by our work with groups of teachers representing grades 6 through 10, so we emphasize habits of thinking that can begin developing in the prealgebra years and, if nurtured, can serve the learning of formal algebra as well. When people think algebraically in order to solve problems, explore, and so on, certain habits of thinking come into play. This chapter discusses three habits that seem to be critical to developing power in algebraic thinking. The list isn't meant to be comprehensive. However, we have no doubt that by learning to attend, in an ongoing fashion, to these several habits—in our own and in students' mathematical work—we will be better prepared to help students succeed in algebra.

A facility with algebraic thinking includes being able to think about *functions* and how they work, and to think about the impact that a system's *structure* has on calculations. These two aspects of algebraic thinking are facilitated by certain habits of mind (Figure 1-1):

- **Doing—Undoing.** Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them). In effect, it is the capacity not only to use a process to get to a goal, but also to understand the process well enough to work backward from the answer to the starting point. So, for example, in a traditional algebraic setting, algebraic thinkers cannot only solve an equation such as $9x^2 - 16 = 0$, but also answer the question, "What is an equation with solutions 4/3 and -4/3?"

FIGURE 1-1. *Three Algebraic Thinking Habits of Mind*



- **Building Rules to Represent Functions.** Critical to algebraic thinking is the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules. For example, here is a functional rule that is computation-based: "Take an input number, multiply it by 4 and subtract 3." This habit of mind is a natural complement to Doing—Undoing, in that the capacity to understand how a functional rule works in reverse generally makes it a more accessible and useful process.
- **Abstracting from Computation.** This is the capacity to think about computations independently of particular numbers that are used. One of the most evident characteristics of algebra has always been its abstractness. But, just what is being abstracted? To answer this, a good case can be made that thinking algebraically involves being able to *think about computations freed from the particular numbers they are tied to in arithmetic*—that is, abstracting system regularities from computation. For example, students invoke this habit of mind when they realize that they can regroup numbers into pairs that equal 101 to make the following computation simpler: "Compute: 1 + 2 + 3 + … + 100." There is a suggestion of Doing—Undoing here, as well, in the recognition that 101 can be decomposed into 100 + 1; 99 + 2; 98 + 3; and so on.

Guiding Questions

Habits of mind develop as the thinker pays attention, over and over again, to "what works" (e.g., what helps in solving problems or what can explain the regularity in a particular pattern) and looks for cues in new situations that previously used approaches may help. Often, the cueing may occur through "guiding questions" that the thinker asks himself. For example, consider these three basic algebraic-thinking questions: How does this process work in reverse? How are things changing in this situation? What are my operation shortcut options to get from here to there? The first and second questions may spur the representation of functions, when that is appropriate; the third may help to spur abstract thinking about calculation, when that is appropriate.

Table 1–1 is a beginning list of guiding questions for each of the three habits of mind. Throughout this book, we try to illustrate the usefulness of the questions with various mathematics activities and examples of student work. The questions in Table 1–1 have developed in our projects over time, and they keep developing and changing as teachers engage with the notion of algebraic-thinking habits of mind and experiment with questions that can help to foster their development in students.

The Role of Classroom Questions

If, as is our belief, these habits of algebraic thinking can be learned, what should teachers be doing to foster the learning? Based on our best information, we can say that productive instruction probably combines the following:

- Consistent modeling of algebraic thinking. For example, in summarizing student responses to a mathematical activity, a teacher might try to make explicit what students have left implicit in their thinking: "So, you decided to try your rule on some larger numbers to see if it would work."
- Giving well-timed pointers to students that help them shift or expand their thinking, or that help them pay attention to what is important. For example, the students of a Linked Learning middle-grades teacher were working on an activity that could be completed satisfactorily via arithmetic. The teacher saw an opportunity for algebraic thinking and said, "Once you have made a chart, look for an easier way. Pay attention to how the numbers group and how the groupings might suggest an easier way."
- Making it a habit to ask a variety of questions aimed at helping students organize their thinking and respond to algebraic prompts. For example, we have noted among some

Linked Learning teachers a consistent use of questions that challenge students to analyze expressions: “Can you explain what the 3 and the 5 represent in that equation?”

TABLE 1-1. *Guiding Questions*

Questions for Doing—and Undoing	**Questions for Building Rules to Represent Functions**	**Questions for Abstracting from Computation**
How is this number in the sequence related to the one that came before?	Is there a rule or relationship here?	How is this calculating situation like/unlike that one?
What if I start at the end?	How does the rule work, and how is it helpful?	How can I predict what’s going to happen without doing all the calculation?
Which process reverses the one I’m using?	Why does the rule work the way it does?	What are my operation shortcut options for getting from here to there?
Can I decompose this number or expression into helpful components?	How are things changing?	When I do the same thing with different numbers, what still holds true? What changes?
	Is there information here that lets me predict what’s going to happen?	What are other ways to write that expression that will bring out hidden meaning?
	Does my rule work for all cases?	How can I write the expression in terms of things I care about?
	What steps am I doing over and over?	How does this expression behave like that one?
	Can I write down a mechanical rule that will do this job once and for all?	
	How can I describe the steps without using specific inputs?	
	When I do the same thing with different numbers, what still holds true? What changes?	
	Now that I have an equation, how do the numbers (parameters) in the equation relate to the problem context?	

We have been paying particular attention to the value of questioning. A couple of beliefs have grown out of the work we have done in our teacher-enhancement projects. One has to do with the role of *intention* in teachers' questioning; the other has to do with the mathematical *context* in which the question is asked:

1. **Intention.** It is valuable for teachers to be aware of the variety and breadth of intention behind classroom questions and to seek, over time, patterns of questioning that are balanced across the range of intention.
2. **Context**. Questions aimed at developing students’ algebraic thinking patterns should be asked in situations that are patently “algebraic,” as well as in situations in which the relevance of algebraic thinking isn't as obvious.

Ideally, once the teacher is able to concentrate wholly on algebraic thinking, the questions will sound like these adaptations from the questions listed in Table 1-1:

- Which process reverses the one you're using?
- How does the rule work?

- How are things changing?
- Can you find a more helpful way to write the rule?
- Is there information here that lets you predict what's going to happen?
- How can you predict what's going to happen without doing all the calculation?
- What are your operation shortcut options for getting from here to there?
- When you do the same thing with different numbers, what still holds true? What changes?

Of course, teachers in real classrooms are dealing with factors that often make the ideal seem remote and, perhaps, unrealistic. Usually, considerable groundwork must be laid for asking algebraic-thinking questions.

Intention

To get a handle on the kinds of questions that lay this groundwork, the Linked Learning Project relied on a cadre of teacher leaders who act as classroom observers of the project's teachers, and asked them to pay special heed to the questions asked by the teachers and the impact of these questions on students. The observations take place in classes in which the teachers are using a lesson meant to elicit algebraic thinking from the students.[1] In post-observation debriefings, the observer checks with the teacher on the accuracy of the observer's judgment about the intended purpose of each recorded question. From the observation data, we found that teachers' questions fall into five categories, according to a teacher's general intention in asking them (Table 1-2).

Any effective lesson or set of lessons will use a blend of question types. Because students can need clarification on what a mathematical challenge is asking them to do, because their attention may need orienting toward key features, and because their underlying reasoning may not be clear or may seem incomplete or faulty, teachers need to use a variety of questions or prompts that are not particularly algebraic, but that lay groundwork for and, ideally, foreshadow students' algebraic thinking.

Context

In addition to teacher intention, mathematical context is another consideration in teachers' use of classroom questions to elicit algebraic thinking and, over time, to foster the development of algebraic habits of mind. Some of the activities that teachers use will display their algebraic potential rather explicitly. For example, there is a clear call for finding a rule to describe a pattern ("What is a general way to say how high the tree will be after *n* months?"), or students are asked to make a general statement drawn from a particular calculation (e.g., "5 = 9 - 4. How many odd numbers can be written as the difference of two perfect squares? Show why you think you have them all."). In such cases, teacher questions can reinforce students' appreciation of what are important features, such as

- comparing the relative value of different representations of a relation: "What does the graph tell you? Now, what different information does the equation give you?"
- looking back, after a solution, for a shortcut that may have been missed and could be useful next time: "How could you have reached that conclusion without looking at the chart?"
- making sure that all the relevant cases have been found: "How can you be sure you have found all the numbers that work?"
- seeing what happens when other numbers are tried: "What if you try a much larger number there?"

TABLE1-2. *Five Categories of Teacher's Questions*

Question Type	**Examples**
Managing Intended to help set students on task, get their work organized, etc.	Who's in charge of writing it down? Are you guys working? What are you doing now?
Clarifying Intended to request information from the student when the teacher isn't clear about what the student means or intends; also, when the teacher is trying to help the student clarify the question	Do you know what *perimeter* is? How did you get 2? (This is asked when the teacher is trying to follow the student's thinking, not trying to correct thinking or help refocus it in a different direction.) Who went first? (during a mathematics game)
Orienting Intended to get students started, or to keep them thinking about the particular problem they are solving; may suggest ways to focus on the problem; also, to orient and/or motivate the student toward the correct answer or away from the incorrect answer	What's the problem asking you to find? Have you thought about trying a table? If you have that number and it increases 3, what do you get? (emphasis on error) How did you get 18 (when the answer is some other value)? How can you check your answer? (wrong answer)
Prompting Mathematical Reflection Intended to ask students to reflect on and explain their thinking; to have them understand others' mathematical ways of thinking; and to have them extend their thinking about the mathematics in a problem	How do you explain that? Can you explain how you got the values in the table? Why did the two of you reach different conclusions? Can you estimate? . . . Now check. Does anyone have a different way?
Eliciting Algebraic Thinking Intended to ask the students to undo, to build rules for describing functional relationships; to abstract from computations they have made; to ask about the meaning of the work they're doing; to ask about what statements are "always" true, about *n*th terms, and about finding patterns and looking for what changes; to work forward and backward, etc.; and to ask students to justify generalizations	What could it (the value in the equation) represent? How could you use the formula? In *x* years, how much does it go up? Can you look for a pattern? Find out how the rule works. What does -2 mean? If this is 13 and this is 16, by how much did it increase? (emphasis on change) What is an easier way? Pay attention to how the numbers group.

In other activities, it is likely that the full algebraic potential—or, in many cases, *any* algebraic potential—will go unexploited unless the teacher asks questions that are used to extend students' thinking about the problem. In those cases, the teacher may be

1. reversing a routine calculating task to challenge students to undo as well as do: "Now that you have a good handle on using a factor tree, answer this: What whole numbers have three factors, including the factor of 1?"
2. asking "what if" questions to extend beyond a single situation to a more generalized situation. For example, suppose students have solved the following problem:

Golden Apples[2]

A prince picked a basketful of golden apples in the enchanted orchard. On his way home, he was stopped by a troll who guarded the orchard. The troll demanded payment of one-half of the apples plus two more. The prince gave him the apples and set off again. A little further on, he was stopped by a second troll guard. This troll demanded payment of one-half of the apples the prince now had plus two more. The prince paid him and set off again. Just before leaving the enchanted orchard, a third troll stopped him and demanded one-half of his remaining apples plus two more. The prince paid him and sadly went home. He had only two golden apples left. How many apples had he picked?

Rather than settle for the solution only, the teacher can further students' thinking by asking, "What if he had 4 left? How many did he begin with? 6 left?", and so on.

3. exploiting calculating situations in which there is regularity, to challenge students to use calculating shortcuts based on the regularity (e.g., "Without writing out all the numbers and adding them, find the total: 1 + 2 + 3 + ... + 27 + 2 8 + 2 7 + . . . + 3 + 2 + 1 ")
4. exploiting calculating situations in which there is regularity, to challenge students to make general statements (e.g., "Think of three consecutive integers and multiply them. Does 2 divide any such product? Why? What other integers divide any such product? What is the largest integer that you can be certain divides any such product evenly? Why?")

Analyses of student work can support the use of classroom questions to foster algebraic thinking, in particular, prompting reflections about appropriate questions. For example, consider Figure 1-2, a piece of student work, drawn from a Linked Learning classroom.

After examining the sample in Figure 1-2, ask yourself what you notice and what it makes you wonder about. What do you infer about the student's line of thinking? What is noticed, inferred, or wondered about can lead to productive instructional questions. What questions might you ask in order to help the student push her thinking further?

One feasible inference is that the student is onto something productive in the last answer, something like, "When I divide the number of the square into the number of its toothpicks, I get 4, 6, 8, 10 for the four squares that I have." From a habit-of-mind perspective, it seems that the student has wondered, "How are things changing?"—a key question in Building Rules to Represent Functions. It also seems that she has done some fluid Doing—Undoing to test how 4, 12, 24, and 40 could be generated from, respectively, 1, 2, 3, 4.

The following are questions that might push the student further: "What have you done to get these numbers?" "When you were counting toothpicks, were you using any counting shortcuts?" "What information is here to help you predict what's going to happen in the next squares?" In "later squares?"